Probability and Random Processes

EES 315

Asst. Prof. Dr. Prapun Suksompong

prapun@siit.tu.ac.th

10 Continuous Random Variables

Office Hours:

Check Google Calendar on the course website.

Dr.Prapun's Office:

6th floor of Sirindhralai building, BKD

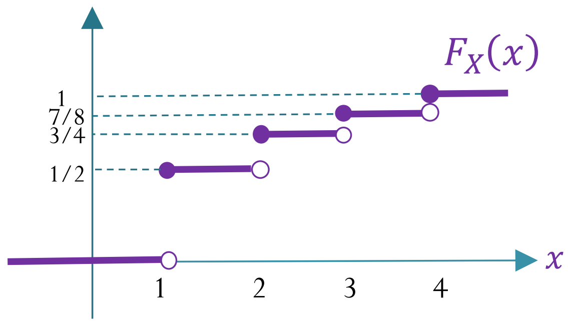
Sections 10.1-10.2

Discrete RV

- **pmf**: $p_X(x) \equiv P[X = x]$
  - Two characterizing properties:
    - $p_X(x) \geq 0$
    - $\sum_x p_X(x) = 1$
- $S_X = \{x: p_X(x) > 0\}$
- $P[\text{some statement(s) about } X]$

$$= \sum_{\substack{\{\text{all the } x \text{ values that} \\ \text{satisfy the statement(s)}\}}} p_X(x)$$

- **cdf** is a staircase function with jumps whose size at $x = c$ gives $P[X = c]$.

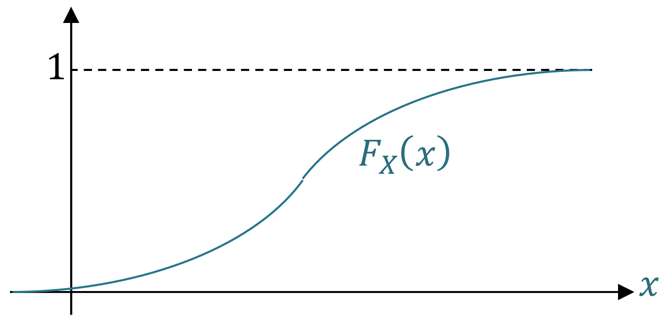


Continuous RV

- $P[X = x] = 0$
- **pdf**: $P[x_0 \leq x \leq x_0 + \Delta x] \approx \overbrace{f_X(x_0)}^{\text{probability per unit length}} \Delta x$
  - Two characterizing properties:
    - $f_X(x) \geq 0$
    - $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
- $S_X = \{x: f_X(x) > 0\}$
- $P[\text{some statement(s) about } X] =$

$$\int_{\substack{\{\text{all the } x \text{ values that} \\ \text{satisfy the statement(s)}\}}} f_X(x) dx$$

- **cdf** is a continuous function.



Chapter 9 vs. Section 10.3

Discrete RV	Continuous RV
$\mathbb{E}X = \sum_{x} x p_X(x)$	$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x) dx$
$\mathbb{E}[g(X)] = \sum_{x} g(x) p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$
$\mathbb{E}[X^2] = \sum_{x} x^2 p_X(x)$	$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$

$$\operatorname{Var}[X] = \mathbb{E}\left[(X - \mathbb{E}X)^2\right] = \mathbb{E}[X^2] - (\mathbb{E}X)^2$$

$$\sigma_X = \sqrt{\operatorname{Var}[X]}$$

Probability and Random Processes

EES 315

Asst. Prof. Dr. Prapun Suksompong

prapun@siit.tu.ac.th

10.1 Probability Density Function

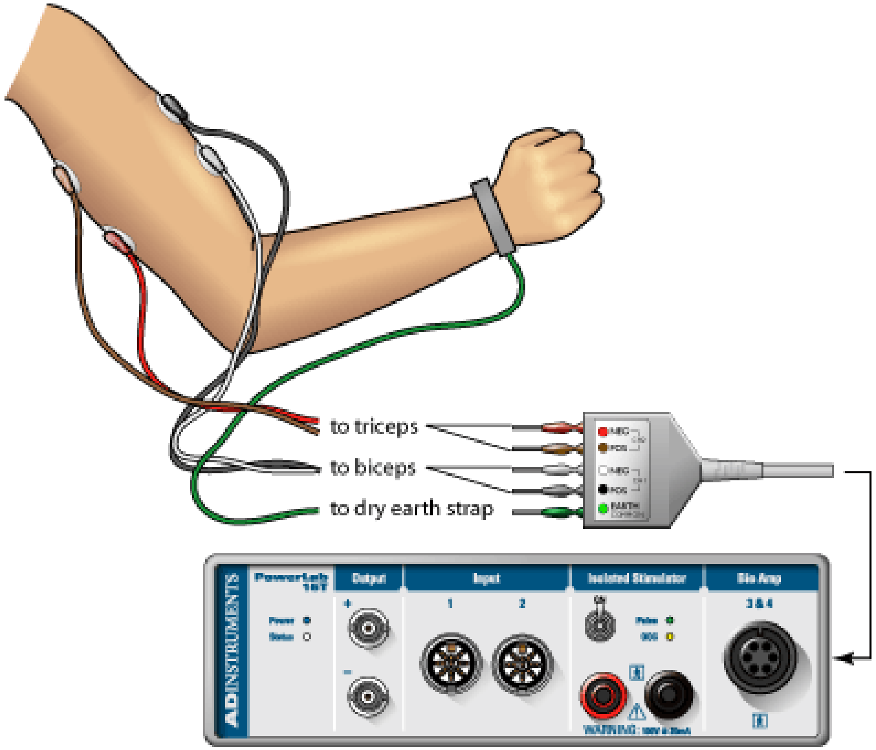
Ex. `rand` function

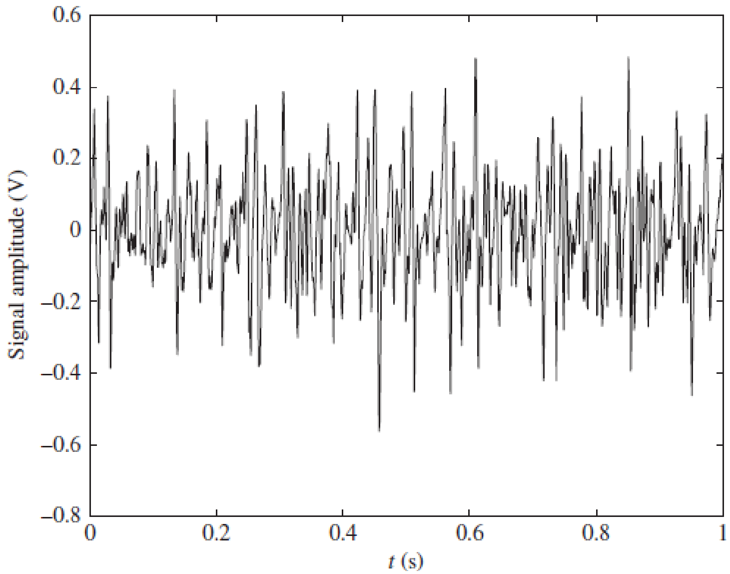
- Generate an array of uniformly distributed pseudorandom numbers.
  - The pseudorandom values are drawn from the **standard uniform distribution** on the open **interval (0,1)**.
- `rand` returns a scalar.
- `rand(m,n)` or `rand([m,n])` returns an *m*-by-*n* matrix.
  - `rand(n)` returns an *n*-by-*n* matrix

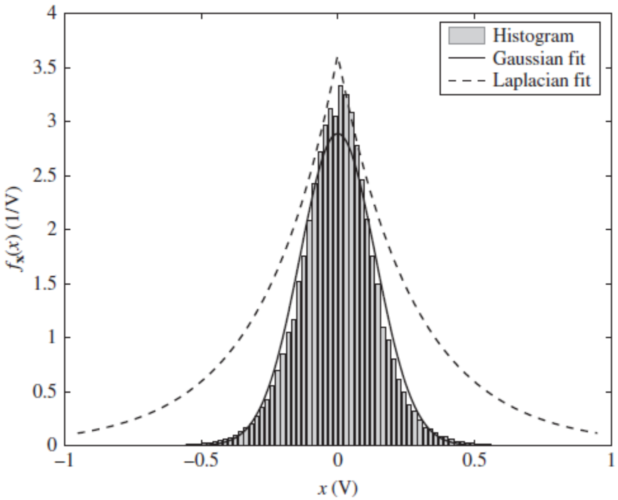
```
>> rand

ans =

    0.3816

>> rand(10,2)

ans =

    0.7655    0.6551
    0.7952    0.1626
    0.1869    0.1190
    0.4898    0.4984
    0.4456    0.9597
    0.6463    0.3404
    0.7094    0.5853
    0.7547    0.2238
    0.2760    0.7513
    0.6797    0.2551
```

Ex. Muscle Activity

- Look at electrical activity of skeletal muscle by recording a human electromyogram (EMG).





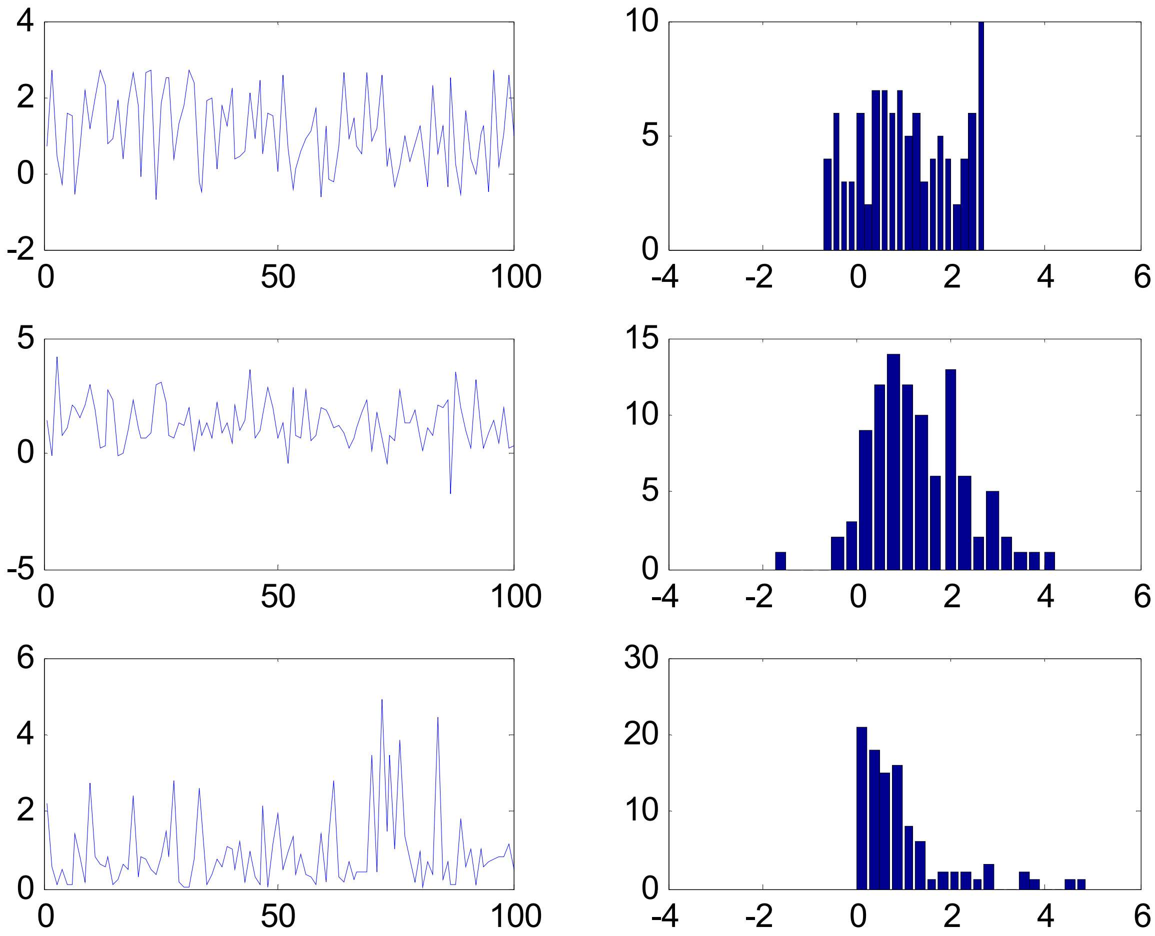


[http://www.adinstruments.com/solutions/education/ltexp/electromyography-emg-german]

Three Important Continuous RVs

```
close all; clear all;
N = 1e6; b = 20; m = 1; s = 1;
R = [1-5*s,1+5*s];
% Uniform
X = (2*sqrt(3)*(rand(1,N)-0.5))+1;
subplot(3,2,1); plot(X);
subplot(3,2,2); plotHistPdf(X,b)
xlim(R)
% Normal
X = randn(1,N)+1;
subplot(3,2,3); plot(X);
subplot(3,2,4); plotHistPdf(X,b)
xlim(R)
% Exponential
X = exprnd(1,1,N);
subplot(3,2,5); plot(X);
subplot(3,2,6); plotHistPdf(X,b)
xlim(R)
```

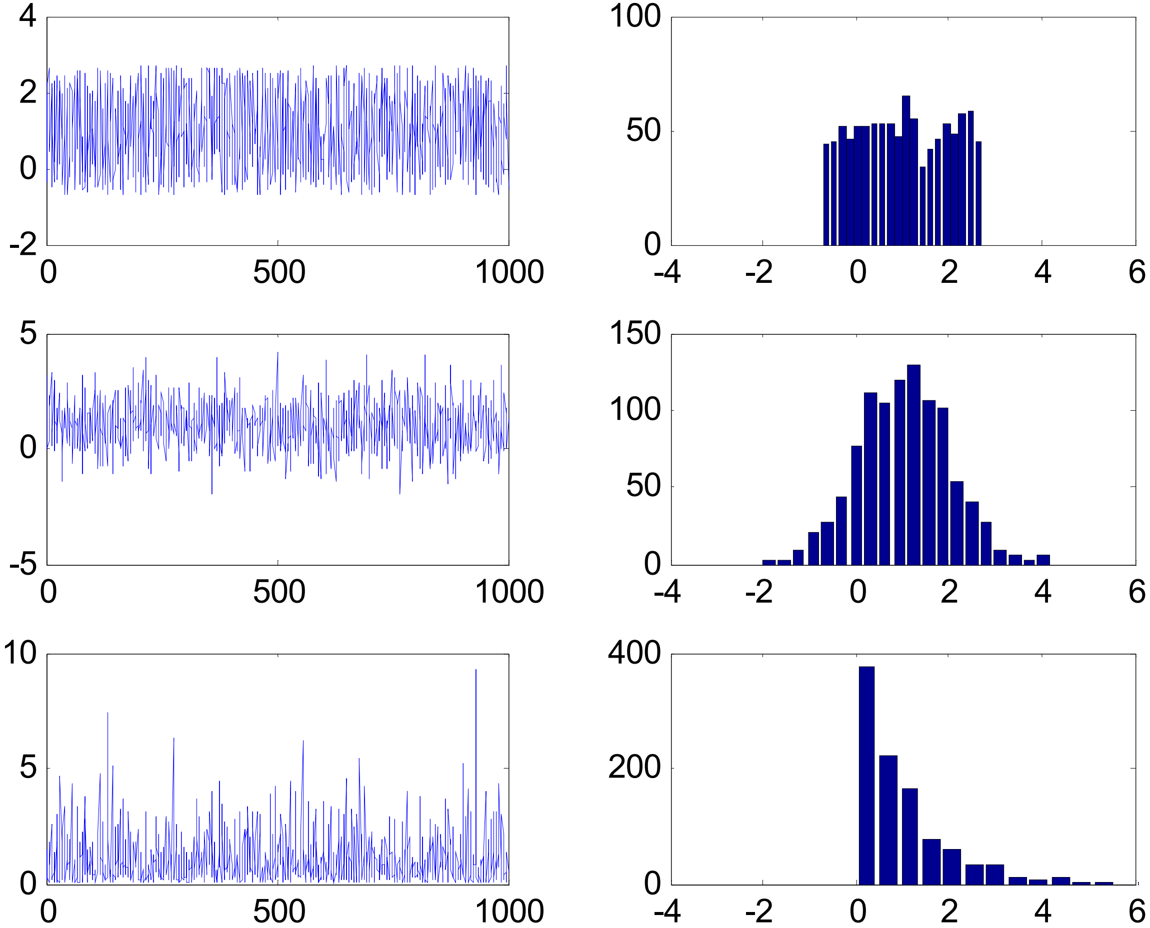
Three Important Continuous RVs



Mean = 1
Std = 1
N = 100

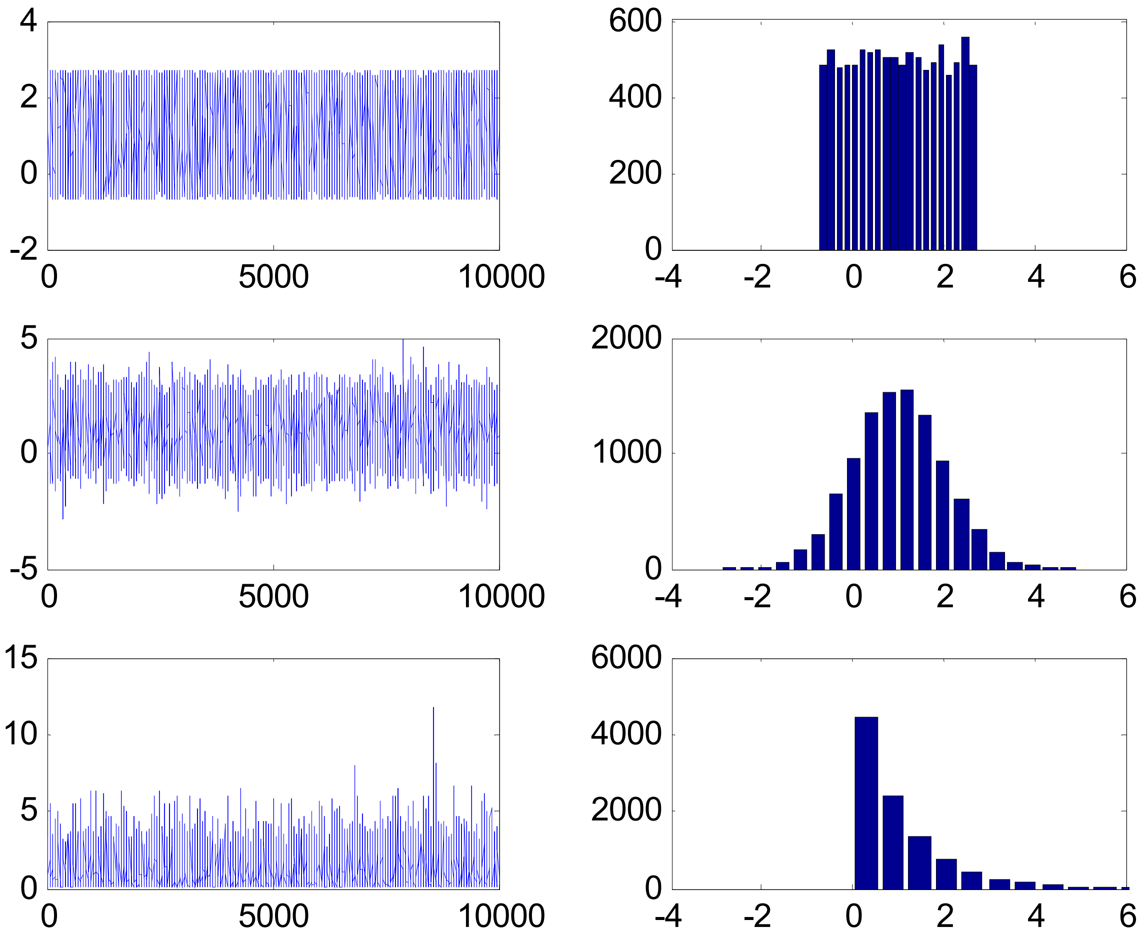
[IntroThreeContinuousRV.m]

Three Important Continuous RVs



Mean = 1
Std = 1
N = 1,000

Three Important Continuous RVs



Mean = 1

Std = 1

N = 10,000

Review: P[some condition(s) on X]

For discrete random variable,

8.14. Steps to find probability of the form P [some condition(s) on X] when the pmf $p_X(x)$ is known.

(a) Find the support of X.

(b) Consider only the x inside the support. Find all values of x that satisfy the condition(s).

(c) Evaluate the pmf at x found in the previous step.

(d) Add the pmf values from the previous step.

$$P[\text{some condition(s) on } X] = \sum p_X(x)$$

Discrete RV

Sum over all the x values that satisfy the condition(s)

P[some condition(s) on X]

- **For discrete random variable,**

$$P[\text{some condition(s) on } X] = \sum \overbrace{p_X(x)}^{\text{probability mass function (pmf)}}$$

Discrete RV

Sum over all the x values that satisfy the condition(s)

- **For continuous random variable,**

pmf → pdf

$\sum \rightarrow \int$

$$P[\text{some condition(s) on } X] = \int \overbrace{f_X(x)}^{\text{probability density function (pdf)}} dx$$

Continuous RV

Integrate over all the x values that satisfy the condition(s)

Support of a RV

- In general, the **support** of a RV X is any set S such that
$$P[X \in S] = 1.$$
- In this class, we try to find the smallest (minimal) set that works as a support.
- **For discrete random variable,**
$$S_X = \{x: p_X(x) > 0\}$$
- **For continuous random variable,**
$$S_X = \{x: f_X(x) > 0\}$$

Probability and Random Processes
EES 315

Asst. Prof. Dr. Prapun Suksompong
prapun@siit.tu.ac.th

10.2 Properties of PDF and CDF

WIKIPEDIA
The Free Encyclopedia

Main page
Contents
Current events
Random article
About Wikipedia
Contact us
Donate

Contribute

Help
Learn to edit
Community portal
Recent changes
Upload file

Tools

What links here
Related changes
Special pages
Permanent link
Page information
Cite this page
Wikidata item

Article Talk

Integral

From Wikipedia, the free encyclopedia

This article is about the concept of definite integrals in calculus. For the indefinite integral, see antide
"Area under the curve" redirects here. For the pharmacology integral, see Area under the curve (pha

In mathematics, an **integral** assigns numbers to functions in a way that can describe displacement, area
infinitesimal data. Integration is one of the two main operations of calculus; its inverse operation, differer
an interval $[a, b]$ of the real line, the **definite integral** of f from a to b can be interpreted informally as th
the graph of f, the x-axis and the vertical lines $x = a$ and $x = b$. It is denoted

$$\int_a^b f(x)\,dx.$$

The operation of integration, up to an additive constant, is the inverse of the operation of differentiation.
notion of the antiderivative, called an **indefinite integral**, a function F whose derivative is the given func

$$F(x) = \int f(x)\,dx.$$

The integrals discussed in this article are those termed *definite integrals*. It is the fundamental theorem c
integral: if f is a continuous real-valued function defined on a closed interval $[a, b]$, then once an antider
interval is given by

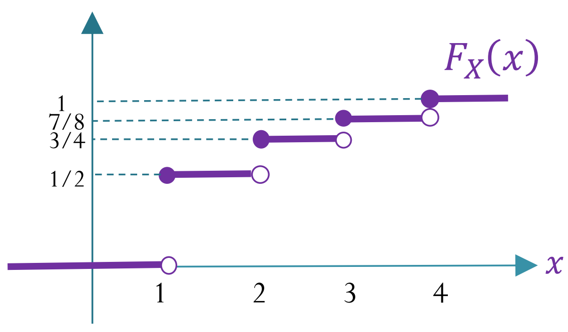
$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)\,.$$

The principles of integration were formulated independently by Isaac Newton and Gottfried Wilhelm Leib

Sections 10.1-10.2

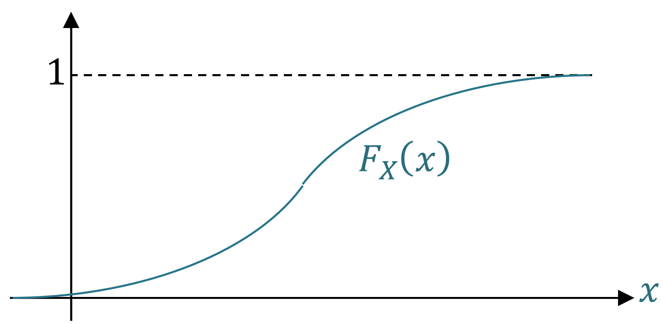
Discrete RV

- **pmf**: $\boldsymbol{p_X}(x) \equiv P[X = x]$
  - Two characterizing properties:
    - $p_X(x) \geq 0$
    - $\sum_x p_X(x) = 1$
- $S_X = \{x: p_X(x) > 0\}$
- $P[\text{some condition(s) on } X]$
  $$= \sum_{\substack{\{\text{all the } x \text{ values that} \\ \text{satisfy the condition(s)}\}}} p_X(x)$$
- **cdf** is a staircase function with jumps whose size at $x = c$ gives $P[X = c]$.



Continuous RV

- $P[X = x] = 0$
- **pdf** : $P[x_0 \leq x \leq x_0 + \Delta x] \approx \overbrace{\boldsymbol{f_X}(x_0)}^{\text{probability per unit length}}\Delta x$
  - Two characterizing properties:
    - $f_X(x) \geq 0$
    - $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
- $S_X = \{x: f_X(x) > 0\}$
- $P[\text{some condition(s) on } X] =$
  $$\int_{\substack{\{\text{all the } x \text{ values that} \\ \text{satisfy the condition(s)}\}}} f_X(x) dx$$
- **cdf** is a continuous function.



pdf and cdf for continuous RV

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
P[a \leq X \leq b] & F_X(x) \equiv P[X \leq x] \arrow[l, "F_X(b) - F_X(a)"'] \arrow[d, shift right=2, "\text{“}\frac{d}{dx}\text{”} F_X(x)"'] \\
& f_X(x) \arrow[ul, "\int_a^b f_X(x)\,dx"] \arrow[u, shift right=2, "\int_{-\infty}^{x} f_X(t)\,dt"']
\end{tikzcd}
$$

$P[a < X < b]$

$P[a \leq X < b]$

$P[a < X \leq b]$

$P[a \leq X \leq b]$

Finding Probabilities from CDF

Definition: $F_X(x) \equiv P[X \le x]$

For **any RV**,	For **continuous RV**,
• $P[X \le b] = F_X(b)$ $P[X < b] = F_X(b) - P[X = b]$	• $P[X \le b] = F_X(b)$ $P[X < b] = F_X(b)$
• $P[X > a] = 1 - F_X(a)$ $P[X \ge a] = 1 - F_X(a) + P[X = a]$	• $P[X > a] = 1 - F_X(a)$ $P[X \ge a] = 1 - F_X(a)$
• $P[a < X \le b] = F_X(b) - F_X(a)$	• $P[a < X \le b] = F_X(b) - F_X(a)$ $P[a < X < b] = F_X(b) - F_X(a)$ $P[a \le X < b] = F_X(b) - F_X(a)$ $P[a \le X \le b] = F_X(b) - F_X(a)$
• $P[X = a] = F_X(a) - F_X(a^-)$ (amount of jump in the CDF @ a)	• $P[X = a] = 0$

Probability and Random Processes
EES 315

Asst. Prof. Dr. Prapun Suksompong
prapun@siit.tu.ac.th

10.3 Expectation and Variance

WolframAlpha computational intelligence.

∫(a*x*exp(-a*x))dx from 0 to infinity

Extended Keyboard | Upload | Examples | Random

Definite integral:

$$\int_0^\infty a\,x\exp(-a\,x)\,dx = \frac{1}{a} \quad \text{for } \operatorname{Re}(a) > 0$$

$\operatorname{Re}(z)$ is the real part of z

∫(a*(x^2)*exp(-a*x))dx from 0 to infinity

Extended Keyboard | Upload | Examples | Random

Definite integral:

$$\int_0^\infty a\,x^2\exp(-a\,x)\,dx = \frac{2}{a^2} \quad \text{for } \operatorname{Re}(a) > 0$$

$\operatorname{Re}(z)$ is the real part of z

Probability and Random Processes

EES 315

Asst. Prof. Dr. Prapun Suksompong

prapun@siit.tu.ac.th

10.4 Families of Continuous Random Variables

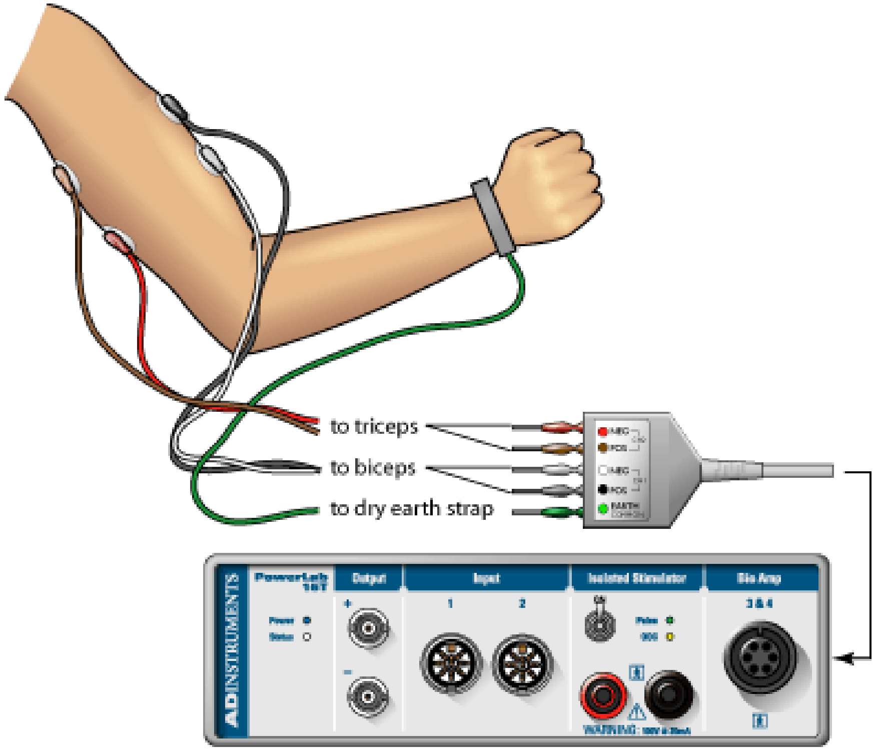
Johann Carl Friedrich Gauss

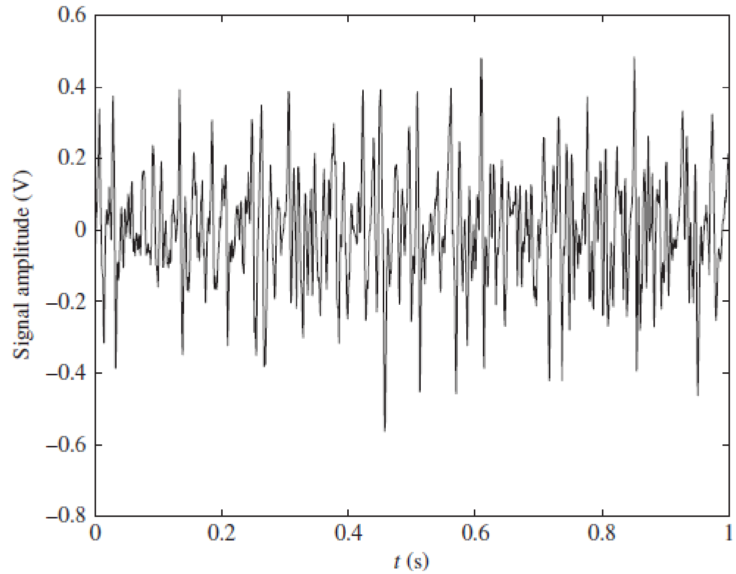
German 10-Deutsche Mark Banknote (1993; discontinued)

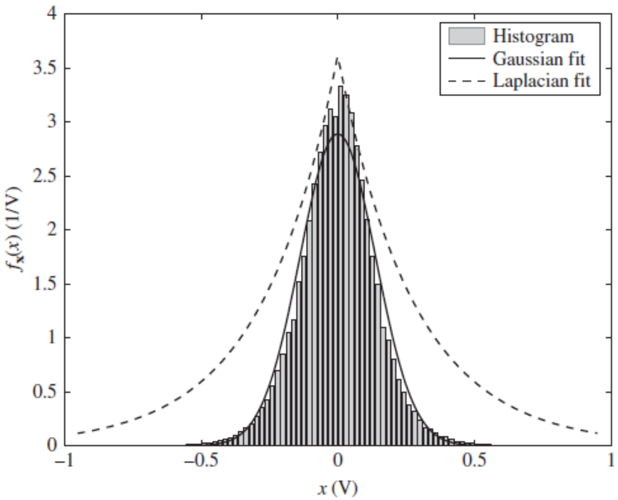
- 1777 –1855
- A German mathematician

Ex. Muscle Activity

- Look at electrical activity of skeletal muscle by recording a human electromyogram (EMG).







[http://www.adinstruments.com/solutions/education/ltexp/electro myography-emg-german]

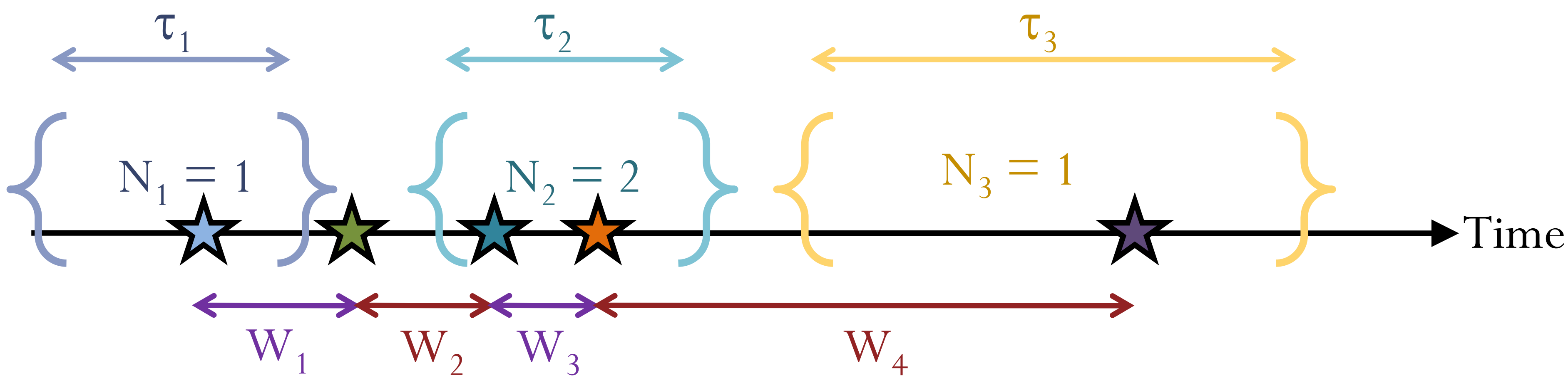
Expected Value and Variance

"Proof" by MATLAB's symbolic calculation

```
>> syms x
>> syms m real
>> syms sigma positive
>> int(1/(sqrt(sym(2)*pi)*sigma)*exp(-(x-m)^2/(2*sigma^2)),x,-inf,inf)
ans =
1
>> EX = int(x/(sqrt(sym(2)*pi)*sigma)*exp(-(x-m)^2/(2*sigma^2)),x,-inf,inf)
EX =
m
>> EX2 = int(x^2/(sqrt(sym(2)*pi)*sigma)*exp(-(x-m)^2/(2*sigma^2)),x,-inf,inf)
EX2 =
-(2^(1/2)*(limit(- x*sigma^2*exp((x*m)/sigma^2 - m^2/(2*sigma^2) - x^2/(2*sigma^2)) - m*sigma^2*exp((x*m)/sigma^2 - m^2/(2*sigma^2) - x^2/(2*sigma^2)) -
(2^(1/2)*pi^(1/2)*sigma*erfi((2^(1/2)*(x - m)*i)/(2*sigma))*(m^2 + sigma^2)*i)/2, x == -Inf) - limit(- x*sigma^2*exp((x*m)/sigma^2 - m^2/(2*sigma^2) -
x^2/(2*sigma^2)) - m*sigma^2*exp((x*m)/sigma^2 - m^2/(2*sigma^2) - x^2/(2*sigma^2)) - (2^(1/2)*pi^(1/2)*sigma*erfi((2^(1/2)*(x - m)*i)/(2*sigma))*(m^2 +
sigma^2)*i)/2, x == Inf)))/(2*pi^(1/2)*sigma)
>> EX2 = simplify(EX2)
EX2 =
m^2 + sigma^2
>> VarX = EX2 - (EX)^2
VarX =
sigma^2
```

Poisson Process

The number of arrivals $N_1, N_2, N_3, \ldots$ during non-overlapping time intervals are independent Poisson random variables with mean $= \lambda \times$ the length of the corresponding interval.



The lengths of time between adjacent arrivals $W_1, W_2, W_3, \ldots$ are i.i.d. exponential random variables with mean $1/\lambda$.